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EXPRESSION AS A LEGENDRE FUNCTION, OF AN ELLIPTIC INTEGRAL OCCURRING IN WING THEORY

by

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ABSTRACT

An integral that often arises in potential theory is shown to be related to half order Legendre functions of the second kind, including a special form of the integral studied by Riegels. Series expansions of these Legendre functions for values of the argument in the neighborhood of unity and much greater than unity are also presented. For convenient reference, recursion formulas and expressions in terms of elliptic integrals are noted.

SYMBOLS

A,B,C	constants
a	real constant
a_{ns}, b_{ns}	coefficients in expansion of $Q_{n-\frac{1}{2}}$
$E(k)$	complete elliptic integral of first kind
$C_n(k^2)$	Riegels' function
$J_\nu(z)$	Bessel function of first kind and order ν
$K(k)$	complete elliptic integral of first kind
k	modulus of elliptic integrals
m	real integer number
n	real integer number
$P_\nu(z)$	Legendre function of first kind
$Q_\nu(z)$	Legendre function of second kind
R	spatial distance between singularity and field point
(x, r, θ)	cylindrical coordinates
$\Pi(n)$	Gauss' function $\Pi(n) = \Gamma(n+1)$
ν	complex number

I. INTRODUCTION

The method of superimposing or distributing singular solutions to Laplace's equation is a powerful way of obtaining solutions to many problems of potential theory.

The introduction of singularities often leads to expressions for the induced field which are inversely proportional to R or R^3 where R is the distance between the singular point and the field point. In hydrodynamics, for instance the velocity induced by a plane source is inversely proportional to R while the use of the Biot-Savart Law leads to expressions containing the inverse cube of R. Using cylindrical coordinates (x,r,θ), we can write the distance R as

$$R = \left\{ (x-\bar{x})^2 + r^2 + \bar{r}^2 - 2r\bar{r}\cos(\theta-\bar{\theta}) \right\}^{\frac{1}{2}} \tag{1}$$

where the bars ($^-$) are used to denote the point of the singularity. Since a trigomonetric Fourier Series expansion in the angle θ is used in many problems

adaptable to polar or cylindrical coordinates, it is of importance to study integrals of the type

$$\int_{-\pi}^{\pi} e^{-im\theta} \frac{f(\cos\theta, \sin\theta)}{R^p} \, d\theta \tag{2}$$

where the function f is analytic and $p=1,3$. Taking into account that the integration interval is symmetric with respect to $\theta=0$ and corresponds to a full period of the integrand, we see that we need, by substitution of $(\theta-\bar{\theta})=2\tau$, to examine essentially integrals of the type

$$\Phi_{np}(a^2) \equiv \int_{-\pi/2}^{\pi/2} \frac{\cos 2n\tau}{(a^2+4\sin^2\tau)^{p/2}} \, d\tau \tag{3}$$

for $p=1,3$. Here n is an integer and a is a real constant.

An integral of similar type has been studied by Riegels[1] who related

$$G_n(k^2) \equiv (-1)^n \int_0^{\pi/2} \frac{\cos 2n\tau}{(1-k^2\sin^2\tau)^{3/2}}\, d\tau \tag{4}$$

for $0 \le k^2 < 1$ to a combination of complete elliptic integrals of the first and second kind. He also gave a complete series representation of $G_n(k^2)$ for small k^2, the first terms of a series valid for k^2 near 1, and tables of $G_n(k^2)$ for $0 \le n \le 7$. $G_n(k^2)$ was encountered by Riegels[2] in a study of the flow past slender, almost axisymmetric bodies, and by J. Weissinger[3] in the aerodynamic theory of a ring airfoil.

The purpose of the present paper is to extend Riegels' work by relating the integrals of Eqs. (3) and (4) to half order Legendre functions of the second kind and to present complete series representations of these functions for values of the argument in the neighborhood of unity and much greater than unity. For convenient reference a set of recursion formulas is included.

II. RELATION TO LEGENDRE FUNCTIONS

The integral Φ_{nl} can be transformed into a double integral by using the Laplace transform of the Bessel function of first kind and order zero, J_0, or

$$\Phi_{nl} = \int_{-\pi/2}^{\pi/2} \cos 2n\tau \, d\tau \quad \times$$

$$\int_0^\infty e^{-|a|t} J_0(2\sin\tau\, t)\, dt \tag{5}$$

The order of integration can be interchanged essentially because the right hand side of Eq. (5) is absolutely convergent if $a \neq 0$. We further observe that J_0 can be expressed as an infinite sum of products of Bessel functions by Neumann s addition theorem[4]. So,

$$\Phi_{nl} = \int_0^\infty e^{-|a|t}\, dt \quad \times$$

$$\int_{-\pi/2}^{\pi/2} (\cos 2n\tau) \left[\sum_{m=0}^{\infty} \epsilon_m J_m^2(t) \cos 2m\tau \right] d\tau \tag{6}$$

where ϵ_m is the Neumann factor:

$$\epsilon_m = \begin{matrix} 2, & m \neq o \\ 1, & m = o \end{matrix}$$

Because the series is absolutely convergent, we can integrate term by term. Since the set of cosine functions is orthogonal in $[-\pi/2, \pi/2]$, only the term for $m=n$ will give a contribution. We finally obtain

$$\Phi_{nl} = \pi \int_0^\infty e^{-|a|t} J_n^2(t)\ dt \tag{8}$$

The Laplace transform that appears in Eq. (8) is expressible as a half order Legendre function of the second kind, $Q_{n-\frac{1}{2}}$. The general formula from p. 389 of Ref. 4 is

$$\int_0^\infty e^{-At} J_\nu(Bt) J_\nu(Ct)\ dt =$$

$$(1/\pi\sqrt{BC})\ Q_{\nu-\frac{1}{2}}\ [(A^2+B^2+C^2)/2BC] \tag{9}$$

where it is assumed that the real parts of all of the four numbers $A \pm iB \pm iC$ are positive and that the real part of ν is greater than $-\frac{1}{2}$.

Applying Eq. (9) to Eq. (8) with $A = a$ and $B = C = 1$ we find that*

$$\int_{-\pi/2}^{\pi/2} \frac{\cos 2n\tau}{(a^2+4\sin^2\tau)^{1/2}}\, d\tau = Q_{n-\frac{1}{2}}(1+a^2/2) \tag{10}$$

Consequently differentiation with respect to the parameter a^2 yields,

$$\int_{-\pi/2}^{\pi/2} \frac{\cos 2n\tau}{(a^2+4\sin^2\tau)^{3/2}}\, d\tau = -Q'_{n-\frac{1}{2}}(1+a^2/2) \tag{11}$$

The prime (') denotes differentiation with respect to the argument of $Q_{n-\frac{1}{2}}$.

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*In the physical case of interest here, a is a real constant. However, Eq. (10) is valid for all a except along the cut from $-\infty i$ to $+i\infty$ in the complex plane.

By setting $\tau = (\pi/2 - t)$ and $k^2 = 4/(a^2+4)$ in Eq. (11), we have upon comparison with Eq. (4)

$$G_n(k^2) = -(4/|k|^3)\, Q'_{n-\frac{1}{2}}(2/k^2-1) \tag{12}$$

Eq. (12) then relates the function G_n, tabulated by Riegels, to the derivative of the half order Legendre functions of the second kind.

III. SERIES REPRESENTATIONS OF $Q_{n-\frac{1}{2}}(z)$

First we consider the case of z near $+1$. The Legendre functions of the second kind have logarithmic singularities when the argument equals ± 1. Since in our application $z=(1+a^2/2)$ and a is a real constant, the singularity at $z=+1$ is of special interest.

A series representation of $Q_{n-\frac{1}{2}}(z)$ valid near $z=1$ can be obtained by assuming a solution of the form

$$Q_{n-\frac{1}{2}}(z) = \sum_{s=0}^{\infty} a_{ns}(z-1)^s + [\ell n(z-1)] \sum_{s=0}^{\infty} b_{ns}(z-1)^s \tag{13}$$

to the corresponding Legendre differential equation,

$$0 = (1-z^2)Q''_{n-\frac{1}{2}}(z) - 2zQ'_{n-\frac{1}{2}}(z) + (n^2-\tfrac{1}{4})Q_{n-\frac{1}{2}}(z) \tag{14}$$

Assume the complex z-plane to be cut from $+1$ to $-\infty$ along the real axis. If Eq. (13) is put into Eq. (14), the following recurrence formulas are obtained by setting the coefficients in front of each power of $(z-1)$ equal to zero:

$$b_{ns+1} = b_{ns}\,\frac{n^2 - \frac{1}{4} - s(s+1)}{2(s+1)^2} \tag{15}$$

$$a_{ns+1} = a_{ns}\,\frac{n^2 - \frac{1}{4} - s(s+1)}{2(s+1)^2} + b_{ns}\,\frac{2(n^2-\frac{1}{4}) + (s+1)}{2(s+1)^3} \tag{16}$$

The first coefficients a_{no} and b_{no} can be obtained from the following expression[5] for $Q_{n-\frac{1}{2}}$:

$$Q_{n-\frac{1}{2}}(z) = \frac{1}{2}P_{n-\frac{1}{2}}(z)\,\ln\frac{z+1}{z-1} + \left[\Pi(0) - \frac{\Pi(n-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\right]P_{n-\frac{1}{2}}(z) + O(z-1) \tag{17}$$

Here $\Pi(n) = \Gamma(n+1)$ is Gauss' function, $P_{n-\frac{1}{2}}$ is a Legendre function of the first kind, and O is the Bachmann-Landau order of magnitude symbol. Using the fact that $P_{n-\frac{1}{2}}(1) = 1$, we can deduce from Eq. (17) that

$$b_{ns} = -\frac{1}{2} \tag{18}$$

$$a_{ns} = \frac{1}{2}\ell n 2 + \left\{ \Pi'(o) - \frac{\Pi'(n-\frac{1}{2})}{\Pi(n-\frac{1}{2})} \right\}$$

$$= \frac{5}{2}\ell n 2 - 2 \sum_{j=1}^{n} \frac{1}{2j-1} \tag{19}$$

Eqs. (13), (15), (16), (18) and (19) provide the desired series expansion of $Q_{n-\frac{1}{2}}(z)$ near $z=+1$. The corresponding expansion for $Q'_{n-\frac{1}{2}}(z)$ is obtained by differentiation of Eq. (13).

Riegels[1] has calculated numerically the coefficients of a series expansion of $G_n(k^2)$ for $k^2 \to 1$. By using Eq. (12), our results can be readily checked with his.

Secondly, we consider the case for large z.
In this case we can use a hypergeometric representation of $Q_{n-\frac{1}{2}}(z)$, for example,

$$Q_{n-\frac{1}{2}}(z) = \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{n!\,2^{n+\frac{1}{2}}} (z+1)^{-n-\frac{1}{2}} \times$$

$$_2F_1\{n+\tfrac{1}{2}, n+\tfrac{1}{2}; 2n+1; 2/(z+1)\} \tag{20}$$

Or, after some rearrangement,

$$Q_{n-\frac{1}{2}}(z) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{\Gamma^2(n+s+\frac{1}{2})}{s!\,(2n+s)!} \{ 2/(z+1) \}^{n+s+\frac{1}{2}} \tag{21}$$

for $z>1$. Riegels' corresponding expansion for $G_n(k^2)$ can be obtained from Eq. (21).

IV. RECURSION FORMULAS FOR $Q_{n-\frac{1}{2}}(z)$ AND $Q'_{n-\frac{1}{2}}(z)$

Recursion formulas have been derived, for example, in Ref. 6, and are given below for convenient reference:

$$Q_{n-\frac{1}{2}}(z) = \frac{1}{2n} \left[Q'_{n+\frac{1}{2}}(z) - Q'_{n-3/2}(z) \right] \tag{22}$$

$$Q'_{n-\frac{1}{2}}(z) = \frac{n-\frac{1}{2}}{z^2-1} \left[zQ_{n-\frac{1}{2}}(z) - Q_{n-3/2}(z) \right] \tag{23}$$

$$Q_{n+\frac{1}{2}}(z) = \frac{2nz}{n+\frac{1}{2}} Q_{n-\frac{1}{2}}(z) - \frac{n-\frac{1}{2}}{n+\frac{1}{2}} Q_{n-3/2}(z) \tag{24}$$

$$Q'_{n+\frac{1}{2}}(z) = \frac{2nz}{n-\frac{1}{2}} Q'_{n-\frac{1}{2}}(z) - \frac{n+\frac{1}{2}}{n-\frac{1}{2}} Q'_{n-3/2}(z) \tag{25}$$

$$Q_{n-\frac{1}{2}}(z) = Q_{-n-\frac{1}{2}}(z) \tag{26}$$

$$Q'_{n-\frac{1}{2}}(z) = Q'_{-n-\frac{1}{2}}(z) \tag{27}$$

The relationship between the half order Legendre functions of the second kind and the complete elliptic integrals of the first and second kind, or $E(k)$ and $K(k)$ respectively, is finally demonstrated by[6]

$$Q_{-\frac{1}{2}}(1+\tfrac{1}{2}a^2) = kK(k) \tag{28}$$

$$Q'_{-\frac{1}{2}}(1+\tfrac{1}{2}a^2) = -\frac{k}{a^2}E(k) \tag{29}$$

where $k=\{4/(a^2+4)\}^{\frac{1}{2}}$. Hence, by using first the recursion formula of Eq. (23) and then that of Eq. (24), it is possible to express the half order Legendre functions in terms of the complete elliptic integrals of the first and second kind for all integer n .

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